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FAST TRACK COMMUNICATION

Geometric phase for non-Hermitian Hamiltonian evolution as anholonomy of a parallel transport along a curve

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Abstract
We develop an interpretation of the geometric phase in evolution with a non-Hermitian real-valued Hamiltonian by relating it to the angle developed during the parallel transport along a closed curve by a unit vector triad in the 3D Minkovsky space. We also show that this geometric phase is responsible for the anholonomy effects in stochastic processes considered by Sinitsyn and Nemenman (2007 *Europhys. Lett.* **77** 58001), and use it to derive the stochastic system response to periodic parameter variations.

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(Some figures in this article are in colour only in the electronic version)

In quantum mechanics, anholonomy effects (i.e. parallel transported vectors not returning to their initial orientations after a motion along a closed curve) usually can be related to the Berry phase [1]. Similar effects have been recognized in many other fields and were also related to several generally defined geometric phases. Examples can be found in classical mechanics [2, 3], hydrodynamics [4], classical chaos [5], soliton dynamics [6], dissipative kinetics [7–9] and stochastic processes [10–15].

Simple systems with a minimal number of degrees of freedom have always been of particular importance. Thus the essential features of the Berry phase in quantum mechanics can be discussed using a two-level system and the corresponding $SU(2)$ group of its evolution. Another simple group of transformations, which was widely discussed in relation to geometric phases, is the $SU(1, 1)$ group, and it is isomorphic to $SL(2, R)$. It is also homomorphic to the Lorentz group $SO(2, 1)$ [16]. The essential difference with respect to the $SU(2)$ case is that the quotient manifold $SU(1, 1)/U(1)$ can be identified with a hyperboloid rather than a sphere. The corresponding geometric phases have been predicted and studied in several classical mechanical [2], relativistic [17–20], quantum mechanical and optical applications [21–24], and were also measured in experiments on polarized light propagation [25–27].

The $SU(2)$ Berry phase anholonomy can be nicely explained by relating it to the rotation angle of a unit vector triad, associated with a closed curve drawn by a unit Bloch vector on a sphere [28–30] (for a textbook demonstration see also [31]). Similar formulation was proved to be useful in other contexts, e.g. in the motion of charged particles in a nonuniform magnetic field [3], light propagation [32] or a motion in a noninertial frame [31]. In [29], it was employed to derive new inequalities for the evolution with the $SU(2)$ group. However, to our knowledge, a similar interpretation of the non-Hermitian $SL(2, R)$ evolution has not been explicitly presented, although it is expected, considering the well-known relation between the $SL(2, R)$ group and the Lorenz group.

In this communication we show exactly how the $SL(2, \mathrm{R})$ geometric phase can be illustrated as the anholonomy of the parallel transport of a vector frame, with a vector triad, defined in the 3D Minkovsky space with correspondingly defined vector multiplication rules. An additional goal is to show that the recently introduced geometric phases in purely classical stochastic kinetics [10, 12] provide one more application of the $SL(2, R)$ geometric phase. We will use this fact to determine the geometric contribution to particle currents in a model proposed in [10], assuming time dependence of all parameters.

Consider the evolution of a real two-state vector $|u\rangle = (u_1, u_2)$ according to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t)\rangle = \hat{H}(t)|u(t)\rangle, \qquad H(t) = \begin{pmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{pmatrix}, \tag{1}$$

with slowly time-dependent real parameters $h_{ij}(t), i, j = 1, 2$. Formally, the solution of (1) can be written as a time-ordered exponent of the time integral of $\hat{H}(t)$,

$$|u(t)\rangle = \hat{U}|u(0)\rangle, \qquad \hat{U} = \hat{T}\left[\mathrm{e}^{\int_0^t \hat{H}(t)\mathrm{d}t}\right]. \tag{2}$$

If the matrix $\hat{H}$ were traceless ($h_{11} = -h_{22}$) the evolution matrix $\hat{U}$ would belong to the group $SL(2, \mathrm{R})$, i.e. the class of 2×2 matrices with real entries and a unit determinant. The requirement to have a zero trace of $\hat{H}$, however, is not crucial for the following discussion, because the nonzero trace merely shifts the eigenvalues of this matrix but does not change its eigenvectors. Therefore the geometric phase is not sensitive to this property, so we will refer to the geometric phase of the group $SL(2, R)$ even if $\hat{H}$ has a nonzero trace.

For the future discussion we will also consider the left state vector $\langle v| = (v_1, v_2)$, evolving according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v| = -\langle v|\hat{H}, \qquad \langle v(t)|u(t)\rangle = 1. \tag{3}$$

Let the matrix $\hat{H}$ have two real eigenvalues. For adiabatically slow evolution of parameters, the right eigenvector corresponding to the larger eigenvalue λ_0 will completely dominate over the other one. If the evolution starts from this eigenvector and parameters pass through a cycle the final vector will return to the initial one; however it will be multiplied by an overall factor e^{ϕ}, i.e.

$$\begin{pmatrix} u_1(T) \\ u_2(T) \end{pmatrix} = \mathrm{e}^{\phi} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix}. \tag{4}$$

The 'phase' ϕ is not imaginary, however, a lot of analogies with quantum-mechanical Berry phase can be established. The Berry phase was generalized to a non-Hermitian evolution [33, 34], and the well-established result is that in the adiabatic limit the phase ϕ can still be written as a sum of dynamical and geometric contributions, i.e. $\phi = \phi_d + \phi_g$, where

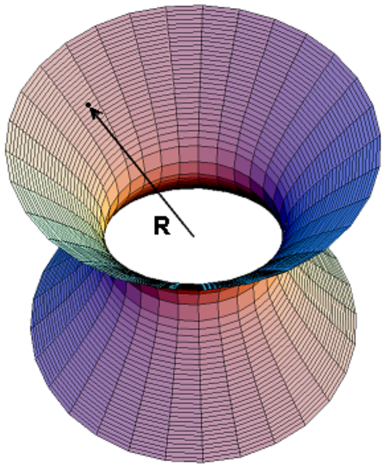

Figure 1. Hyperboloid representing possible states of the vector **R**.

$\phi_d = \int_0^T \lambda_0(t)\,\mathrm{d}t$. The expression for the geometric phase can be written as a parallel transport condition. For this one should redefine states $|u(t)\rangle \rightarrow \mathrm{e}^{-\int_0^t \lambda_0(t)\mathrm{d}t}|u(t)\rangle$, and $\langle v(t)| \rightarrow \mathrm{e}^{\int_0^t \lambda_0(t)\mathrm{d}t}\langle v(t)|$. The geometric phase ϕ_g can then be expressed as arising from the condition [30]

$$\langle v(t)|\partial_t u(t)\rangle = 0, \tag{5}$$

i.e. if we assume that $|u(t)\rangle = \mathrm{e}^{\phi_g(t)}|u(\{h_{ij}\}\rangle$, and $\langle v(t)| = \mathrm{e}^{-\phi_g(t)}\langle v(\{h_{ij}\}|$, where $|u(\{h_{ij}\}\rangle$, and $\langle v(\{h_{ij}\}|$ are instantaneous gauge-fixed normalized right and left eigenstates, corresponding to the same eigenvalue $\lambda_0(\{h_{ij}\})$ of the matrix $\hat{H}$, then the geometric phase after completion of the cyclic evolution reads

$$\phi_g = -\oint \mathrm{d}t\,\langle v(\{h_{ij}\})|\partial_t u(\{h_{ij}\})\rangle. \tag{6}$$

Urbantke [30] showed that for a quantum-mechanical spin-1/2 the condition analogous to (5) has a simple geometrical interpretation in terms of a parallel transport of a unit vector triad. Now we show that a similar interpretation is possible for the $SL(2, R)$ group, however, the triad should be defined in the 3D Minkovsky space.

Components of the right and left vectors $|u\rangle$ and $\langle v|$ can be used to compose a vector **R** such as

$$\mathbf{R} = (x, y, z) = (u_1v_1 - u_2v_2, v_2u_1 + v_1u_2, v_2u_1 - v_1u_2). \tag{7}$$

The normalization condition in (3) then leads to the following normalization of **R**,

$$\mathbf{R} \,\tilde{\cdot}\, \mathbf{R} = x^2 + y^2 - z^2 = 1, \tag{8}$$

where we introduced a scalar product operation in the 3D Minkovsky space $\mathbf{a} \,\tilde{\cdot}\, \mathbf{b} \equiv a_1b_1 + a_2b_2 - a_3b_3$. Figure 1 shows that vector **R** can be represented by a point on a unit hyperboloid immersed in the 3D Minkovsky space. Let us introduce

$$\mathbf{P} = \left(-2u_1u_2, u_1^2 - u_2^2, u_1^2 + u_2^2\right), \qquad \mathbf{Q} = \left(-2v_1v_2, v_1^2 - v_2^2, -\left(v_1^2 + v_2^2\right)\right), \tag{9}$$

and compose two more vectors out of them,

$$\mathbf{N} = (\mathbf{P} + \mathbf{Q})/2, \qquad \mathbf{S} = (\mathbf{P} - \mathbf{Q})/2. \tag{10}$$

One can check that **R**, **N** and **Q** are mutually orthogonal with respect to the metric $(+,+,-)$, namely

$$\begin{aligned} &\mathbf{R}\,\tilde{\cdot}\,\mathbf{S} = \mathbf{R}\,\tilde{\cdot}\,\mathbf{N} = \mathbf{S}\,\tilde{\cdot}\,\mathbf{N} = 0,\\ &\mathbf{R}\,\tilde{\cdot}\,\mathbf{R} = \mathbf{N}\,\tilde{\cdot}\,\mathbf{N} = 1,\\ &\mathbf{S}\,\tilde{\cdot}\,\mathbf{S} = -1. \end{aligned} \tag{11}$$

Vectors **R**, **N** and **S** comprise a unit triad in the 3D Minkovsky space, such that vectors **R** and **N** are spacelike and **S** is timelike.

What does the parallel transport condition (5) mean for the evolution of the triad? For the components of $|u\rangle$ and $\langle v|$ this means that $v_1\partial_t u_1 + v_2\partial_t u_2 = 0$. Following Urbantke [30] this suggests that $\partial_t u_1 = -\lambda_1 v_2$, $\partial_t v_1 = -\lambda_2 u_2$, $\partial_t u_2 = \lambda_1 v_1$ and $\partial_t v_2 = \lambda_2 u_1$, with some variables λ_1 and λ_2 that depend on the details of the evolution Hamiltonian. Substituting this into the definition of the triad vectors we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{N}\\ \mathbf{R}\\ \mathbf{S}\end{pmatrix} = \begin{pmatrix}0 & \tau & 0\\ -\tau & 0 & \varkappa\\ 0 & \varkappa & 0\end{pmatrix}\begin{pmatrix}\mathbf{N}\\ \mathbf{R}\\ \mathbf{S}\end{pmatrix}, \qquad \tau = -(\lambda_1+\lambda_2), \quad \varkappa = \lambda_2 - \lambda_1. \tag{12}$$

Conditions (12) have the form of Serret–Frenet equations in 3D Minkovsky spacetime. According to [35] they describe a unique regular curve parametrized by t, with a curvature $\varkappa$ and torsion τ. From $\dot{\mathbf{N}} = \tau\mathbf{R}$ and $\mathbf{N}\cdot\mathbf{R} = 0$, it follows that $\tau = \dot{\mathbf{N}}\,\tilde{\cdot}\,\mathbf{R} = -\dot{\mathbf{R}}\,\tilde{\cdot}\,\mathbf{N}$, and a similar relation holds for $\varkappa$ in terms of **S** and $\dot{\mathbf{R}}$, which, substituted back in (12), results in

$$\dot{\mathbf{N}} = -(\mathbf{N}\,\tilde{\cdot}\,\dot{\mathbf{R}})\mathbf{R}, \qquad \dot{\mathbf{S}} = -(\mathbf{S}\,\tilde{\cdot}\,\dot{\mathbf{R}})\mathbf{R}. \tag{13}$$

This type of vector evolution is a particular case of the Fermi–Walker vector transport in special relativity, playing an important role in the theory of the Thomas precession [3, 20]. It can be interpreted as follows. Suppose that vectors **N** and **S** at point $\mathbf{R}(t+\mathrm{d}t)$ are obtained by translation of vectors $\mathbf{N}(\mathbf{R}(t))$ and $\mathbf{S}(\mathbf{R}(t))$ to the point $\mathbf{R}(t+\mathrm{d}t)$ that is followed by a projection onto the 2D subspace of vectors orthogonal to $\mathbf{R}(t+\mathrm{d}t)$. Up to higher order in $\mathrm{d}t$, one can write $\mathbf{N}(t)\,\tilde{\cdot}\,\mathbf{R}(t+\mathrm{d}t) \approx \mathbf{N}\,\tilde{\cdot}\,\dot{\mathbf{R}}(t)\,\mathrm{d}t$, and a similar relation holds for **S**. Then $\mathbf{N}(t+\mathrm{d}t) = \mathbf{N}(t) - \mathbf{N}(t)\,\tilde{\cdot}\,\dot{\mathbf{R}}(t)\,\mathrm{d}t$, and $\mathbf{S}(t+\mathrm{d}t) = \mathbf{S}(t) - \mathbf{S}(t)\,\tilde{\cdot}\,\dot{\mathbf{R}}(t)\,\mathrm{d}t$. This means that conditions (5) and (13) correspond to the parallel transport of vectors **N** and **S** along the curve $\mathbf{R}(t)$.

Parallel transported vectors generally do not return to the initial ones after a motion along a closed curve, which represents the anholonomy effect. The relation of such an anholonomy to the geometric phase can be inferred if we observe how these vectors change under the gauge transformations $|u'\rangle = \mathrm{e}^{\phi}|u\rangle$ and $\langle v'| = \langle v|\,\mathrm{e}^{-\phi}$. This corresponds to $\mathbf{P}' = \mathbf{P}\,\mathrm{e}^{2\phi}$ and $\mathbf{Q}' = \mathbf{Q}\,\mathrm{e}^{-2\phi}$. The triad transformation then reads

$$\begin{aligned} &\mathbf{R}' = \mathbf{R},\\ &\mathbf{N}' = \mathbf{N}\cosh(2\phi) + \mathbf{S}\sinh(2\phi),\\ &\mathbf{S}' = \mathbf{N}\sinh(2\phi) + \mathbf{S}\cosh(2\phi), \end{aligned} \tag{14}$$

which indicates that the vector **R** is gauge invariant, but the vectors **N** and **S** are mixed with each other like after a boost transformation in the Minkovsky space. The normalization properties (11), however, remain unaltered. This result means that if after the parallel transport along a closed curve the vectors **N** and **S** become mixed with the angle ϕ, it corresponds to a multiplication of the state vector $|u\rangle$ by an exponential geometric phase factor $\exp(\phi_g)$, where

$$\phi_g = \phi/2. \tag{15}$$

To derive the geometric phase, it is thus sufficient to compare the rotation of the parallel transported vectors **N** and **S** to a pair of fixed reference vectors. Let us introduce a vector product operation $(\mathbf{a}\,\tilde{\times}\,\mathbf{b})_i = g_{ik}\epsilon^{ksm}a_s b_m$, where g_{ik} is the metric tensor of the 3D Minkovsky space with signature (+, +, −), and ϵ^{ksm} is the Levy-Civita symbol. It is possible to assign the fixed triad field $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ in the Minkovsky space as follows:

$$\mathbf{e_3} = (0, 0, 1), \qquad \mathbf{e_1} = \frac{\mathbf{R}\,\tilde{\times}\,\mathbf{e_3}}{|\mathbf{R}\,\tilde{\times}\,\mathbf{e_3}|}, \qquad \mathbf{e_2} = \frac{\mathbf{R}\,\tilde{\times}\,\mathbf{e_1}}{R}, \tag{16}$$

where $R = \sqrt{x^2 + y^2 - z^2}$. Explicitly,

$$\mathbf{e_1} = \left(\frac{y}{\sqrt{x^2+y^2}}, \frac{-x}{\sqrt{x^2+y^2}}, 0\right), \qquad \mathbf{e_2} = \frac{1}{R}\left(\frac{xz}{\sqrt{x^2+y^2}}, \frac{yz}{\sqrt{x^2+y^2}}, \sqrt{x^2+y^2}\right). \tag{17}$$

It is straightforward to show that

$$\begin{aligned} &\mathbf{R}\,\tilde{\cdot}\,\mathbf{e_1} = \mathbf{R}\,\tilde{\cdot}\,\mathbf{e_2} = \mathbf{e_1}\,\tilde{\cdot}\,\mathbf{e_2} = 0, \\ &\mathbf{e_1}\,\tilde{\cdot}\,\mathbf{e_1} = -\mathbf{e_2}\,\tilde{\cdot}\,\mathbf{e_2} = 1. \end{aligned} \tag{18}$$

Consequently, $\mathbf{e_1}$ and $\mathbf{e_2}$ provide a pair of orthogonal unit vectors in the space orthogonal to $\mathbf{R}$. Vector $\mathbf{e_1}$ is spacelike and $\mathbf{e_2}$ is timelike. Note that although the corresponding vector fields are fixed, the local frame $\mathbf{e_1}(\mathbf{R}(t))$ and $\mathbf{e_2}(\mathbf{R}(t))$ will depend on t for an observer, moving along a trajectory $\mathbf{R}(t)$. During the parallel transport the pair $\mathbf{N}$, $\mathbf{S}$ would also rotate around $\mathbf{e_1}$, $\mathbf{e_2}$:

$$\begin{pmatrix}\mathbf{N}(t)\\ \mathbf{S}(t)\end{pmatrix} = \begin{pmatrix}\cosh(\phi(t)) & \sinh(\phi(t))\\ \sinh(\phi(t)) & \cosh(\phi(t))\end{pmatrix}\begin{pmatrix}\mathbf{e_1}(\mathbf{R}(t))\\ \mathbf{e_2}(\mathbf{R}(t))\end{pmatrix}. \tag{19}$$

From the parallel transport conditions, it follows that

$$\mathbf{N}\,\tilde{\cdot}\,\mathrm{d}\mathbf{S} = 0. \tag{20}$$

Substituting (19) into (20) and then using (17) we find that this leads to

$$2\,\mathrm{d}\phi_g = \mathrm{d}\phi = -\mathbf{e_1}\,\tilde{\cdot}\,\mathrm{d}\mathbf{e_2} = -\frac{zy\,\mathrm{d}x - zx\,\mathrm{d}y}{R(x^2+y^2)}. \tag{21}$$

The geometric phase acquired after the motion of vector $\mathbf{R}$ along a closed contour can then be written as

$$\phi_g = \oint_{\mathbf{c}} \mathbf{A}\cdot\mathrm{d}\mathbf{R} = \iint_{S_{\mathbf{c}}} F, \tag{22}$$

where $\mathbf{A} = \left(-\frac{zy}{2R(x^2+y^2)}, \frac{zx}{2R(x^2+y^2)}, 0\right)$, and in the last step we used the Stokes theorem to express a contour integral along $\mathbf{c}$ as an integral over the surface $S_{\mathbf{c}}$ inside this contour from the Berry curvature. The latter, on the surface of the unit hyperboloid ($R = 1$), explicitly reads

$$F = -\tfrac{1}{2}(x\,\mathrm{d}y\wedge\mathrm{d}z + y\,\mathrm{d}z\wedge\mathrm{d}x + z\,\mathrm{d}x\wedge\mathrm{d}y). \tag{23}$$

This curvature 2-form is well known in relation to the groups $SU(1,1)$ and $SL(2,R)$ [23]. Our derivation, however, presents a simple illustration of the geometric origin of this Berry curvature.

To switch from the integration over the surface inside $\mathbf{R}(t)$ to the integral over the surface in the parameter space $\{h_{ij}\}$, note that the vector $\mathbf{R}$ satisfies the Bloch equation [25]

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \xi\,\tilde{\times}\,\mathbf{R}, \tag{24}$$

where

$$\xi(\{h_{ij}\}) = -(h_{11} - h_{22}, h_{12} + h_{21}, h_{12} - h_{21}). \tag{25}$$

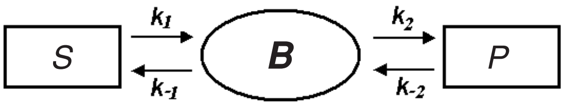

Figure 2. Transition rates into and out of the absorbing states S (substrate) and P (product) through an intermediate bin B-system. The bin can have only zero or one particle inside it.

The quasi-steady-state solution corresponds to

$$\mathbf{R}(\{h_{ij}\}) = -\xi(\{h_{ij}\})/|\xi(\{h_{ij}\})|. \tag{26}$$

As an example of a new application of $SL(2, R)$ formalism, we consider the geometric phase that was found in a purely classical stochastic system. The authors of [10] analyzed stochastic particle fluxes from right to left reservoirs through an intermediate bin system with exclusion interactions, i.e. allowing at most one particle to be inside the bin. Kinetic rates are shown in figure 2. The moments generating function of the particle current is defined as[3]

$$Z(\chi, t) = \mathrm{e}^{S(\chi,t)} = \sum_{n=-\infty}^{\infty} P_n \,\mathrm{e}^{n\chi}, \tag{27}$$

where P_n is the probability of finding a total of n particles transferred from left to right during the observation time t. The authors of [10] showed that (27) can be expressed as the average of the evolution operator

$$Z(\chi, t) = \mathbf{1}^{+}\hat{T}\left(\mathrm{e}^{-\int_0^t \hat{H}(\chi,t)\mathrm{d}t}\right)\mathbf{p}(0), \tag{28}$$

where

$$\hat{H}(\chi, t) = \begin{pmatrix} k_1 + k_{-2} & -k_{-1} - k_2\,\mathrm{e}^{\chi} \\ -k_1 - k_{-2}\,\mathrm{e}^{-\chi} & k_{-1} + k_2 \end{pmatrix}, \tag{29}$$

$\mathbf{1}^{+} = (1, 1)$, and $\mathbf{p}(0) = (p_1, p_2)$ is the vector of initial probabilities of the bin states. Up to a matrix proportional to the unit one, the matrix $\hat{H}(\chi, t)$ in (29) belongs to a set of generators of the $SL(2, R)$ group, thus allowing us to apply all known results for this group to expression (28).

Suppose, parameters k_1 and k_{-2} evolve around a closed contour. From the above discussion it follows that after completing the cycle, the moments generating function becomes an exponent of the sum of two terms:

$$Z(\chi) = \mathrm{e}^{S_{\mathrm{geom}}(\chi)+S_{qst}(\chi)}, \tag{30}$$

where $S_{qst}(\chi)$ is the quasistationary cumulants generating function averaged over all parameter values along the contour, and S_{geom} is the geometric phase contribution responsible for additional pump currents. It can be written as an integral over the surface inside the contour created by the curve in the parameter space,

$$S_{\mathrm{geom}}(\chi) = \int\!\!\int_{S_c} F_{k_1,k_{-2}}\,\mathrm{d}k_1\,\mathrm{d}k_{-2}. \tag{31}$$

[3] In [10] the parameter χ is defined to be multiplied by an imaginary unit $\chi \to \mathrm{i}\chi$. It is, however, clear that making it imaginary played a purely decorative role in the discussion, so we switch to the description of the model using only real parameters in order to highlight the relation to the $SL(2, R)$ group.

Having the general result for the $SL(2, R)$ group (22), it is now straightforward to find the Berry curvature in (31) by a simple change of variables, i.e.

$$F_{k_1,k_{-2}}(\mathbf{k}) = -\frac{1}{2}\left[x(\mathbf{k})\frac{\partial(y,z)}{\partial(k_1,k_{-2})} + y(\mathbf{k})\frac{\partial(z,x)}{\partial(k_1,k_{-2})} + z(\mathbf{k})\frac{\partial(x,y)}{\partial(k_1,k_{-2})}\right]$$
$$= \frac{e_{-\chi}(\mathrm{e}^{\chi}k_2 + k_{-1})}{[4\kappa_+e_\chi + 4\kappa_-e_{-\chi} + K^2]^{3/2}}, \qquad (32)$$

where the components of $\mathbf{R}$ were taken from (25) and (26), $\kappa_\pm \equiv k_{\pm1}k_{\pm2}$, $e_{\pm\chi} \equiv \mathrm{e}^{\pm\chi} - 1$, $K \equiv \sum_m k_m$. The Berry curvature in (32) is the same as the one derived in [10]. It is now easy to derive other previously unknown components of the Berry curvature tensor by a similar change of variables:

$$F_{k_1,k_2}(\mathbf{k}) = -\frac{e_\chi(k_{-1} - k_{-2})}{[4\kappa_+e_\chi + 4\kappa_-e_{-\chi} + K^2]^{3/2}}, \qquad F_{k_2,k_{-2}}(\mathbf{k}) = -\frac{e_{-\chi}(\mathrm{e}^{\chi}k_1 + k_{-1})}{[4\kappa_+e_\chi + 4\kappa_-e_{-\chi} + K^2]^{3/2}},$$
$$F_{k_1,k_{-1}}(\mathbf{k}) = -\frac{e_{-\chi}(\mathrm{e}^{\chi}k_2 + k_{-2})}{[4\kappa_+e_\chi + 4\kappa_-e_{-\chi} + K^2]^{3/2}}, \qquad F_{k_{-1},k_{-2}}(\mathbf{k}) = \frac{e_{-\chi}(k_2 - k_1)}{[4\kappa_+e_\chi + 4\kappa_-e_{-\chi} + K^2]^{3/2}}, \qquad (33)$$
$$F_{k_{-1},k_2}(\mathbf{k}) = -\frac{e_{-\chi}(\mathrm{e}^{\chi}k_1 + k_{-2})}{[4\kappa_+e_\chi + 4\kappa_-e_{-\chi} + K^2]^{3/2}}.$$

In conclusion, we demonstrated that, by analogy to the $SU(2)$ group, the anholonomy of the $SL(2, R)$ evolution can also be illustrated as a rotation of a parallel transported triad along a curve, but in the 3D Minkovsky space. Several theoretical results for the $SU(2)$ group have been derived using such an interpretation [29], and one can attempt to derive similar expressions for the non-Hermitian evolution, however, we do not pursue them here. Instead, we pointed out that the model of a stochastic pump, developed in [10], leads to an evolution described by the $SL(2, R)$ group, and we used it to derive all components of the Berry curvature in the parameter space. Our work should help a further understanding of the stochastic pump effect. For example, the non-adiabatic extension of the $SL(2, R)$ geometric phase has been studied previously [36]. It should be possible to transfer some of the results of that study to the problem of driven stochastic transport, and thus extend the recent progress on the stochastic pump effect in the non-adiabatic regime [13]. It would also be important to find whether the quantization of stochastic pump currents [37] can be related to topological properties of the underlying symmetry group of evolution of the moments generating function.

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References

[1] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[2] Hannay J H 1985 *J. Phys. A: Math. Gen.* **18** 221
[3] Littlejohn R J 1988 *Phys. Rev.* A **38** 6034
[4] Shapere A and Wilczek F 1987 *Phys. Rev. Lett.* **58** 2051
Shapere A and Wilczek F 1988 *J. Fluid Mech.* **198** 557
[5] Jarzynski C 1995 *Phys. Rev. Lett.* **74** 1732
[6] Alber M S and Marsden J E 1992 *Commun. Math. Phys.* **149** 217
[7] Kagan M L, Kepler T B and Epstein I R 1991 *Nature* **349** 506
Kepler T B and Kagan M L 1991 *Phys. Rev. Lett.* **66** 847
[8] Landsberg A S 1992 *Phys. Rev. Lett.* **69** 865
[9] Sinitsyn N A and Ohkubo J 2008 *J. Phys. A: Math. Theor.* **41** 262002

[10] Sinitsyn N A and Nemenman I 2007 *Europhys. Lett.* **77** 58001 (*Preprint* q-bio/0612018)
[11] Sinitsyn N A and Nemenman I 2007 *Phys. Rev. Lett.* **99** 220408
[12] Sinitsyn N A 2007 *Phys. Rev.* B **76** 1
[13] Ohkubo Jun 2008 *J. Stat. Mech.* P02011
[14] Shi Y and Niu Q 2002 *Europhys. Lett.* **59** 324
[15] Astumian D 2007 *Proc. Natl Acad. Sci. USA* **104** 19715
[16] Azcarraga J A and Izquierdo J M 1995 *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics* (Cambridge: Cambridge University Press)
[17] Mukunda N, Aravin P K and Simon R 2003 *J. Phys. A: Math. Gen.* **36** 2347
[18] Ferraro R and Thibeault M 1999 *Eur. J. Phys.* **20** 143
[19] Han D, Kim Y S and Noz M E 1999 *Phys. Rev.* E **60** 1036
[20] Han D, Hardekopf E E and Kim Y S 1989 *Phys. Rev.* A **39** 1269
[21] Klyshko D N 1993 *Usp. Fiz. Nauk* **163** 1
[22] Jordan T F 1988 *J. Math. Phys.* **29** 2042
[23] Vinet L 1988 *Phys. Rev.* D **37** 2369
[24] Benedek C and Beenedict M G 1997 *Europhys. Lett.* **39** 347
[25] Kitano M 1995 *Phys. Rev.* A **51** 4427
[26] Gerry Ch C 1989 *Phys. Rev.* A **39** 3204
[27] Kitano M and Yabuzaki T 1989 *Phys. Lett.* A **142** 321
[28] Dandoloff R and Zakrzewski W J 1989 *J. Phys. A: Math. Gen.* **22** L461
[29] Dandoloff R, Balakrishnan R and Bishop A R 1992 *J. Phys. A: Math. Gen.* **25** L1105
[30] Urbantke H 1991 *Am. J. Phys.* **59** 503
[31] Chruscinski D and Jamiolkowski A 2004 *Geometric Phases in Classical and Quantum Mechanics* (Boston: Birkhäuser)
[32] Berry M V 1987 *Nature* **326** 277
[33] Dattoli G, Mignani R and Torre A 1990 *J. Phys. A: Math. Gen.* **23** 5795
[34] Berry M V 1990 *Proceed. Math. Phys. Sci.* **430** 405
[35] Formiga J B and Romero C 2006 *Am. J. Phys.* **74** 1012
[36] Gao X-C, Xu J-B and Qian T-Z 1992 *Phys. Rev.* A **46** 3626
[37] Dean Astumian R and Derényi I 2001 *Phys. Rev. Lett.* **86** 3859